Dynamics of a Floating Ice Sheet

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Civilian and military operations are frequently conducted on the surface of floating ice sheets. For reasons of safety and operational reliability it is often required to predict the deformation and stresses in such floating ice plates due to surface loading. The present dynamic analysis considers the deformation of a floating ice sheet of infinite extent subjected to rapidly applied surface loads. Solutions are obtained within the framework of both improved and classical theories.

Nomenclature

Dimensionless quantity	To convert to dimensional form multiply by	Physical interpretation
r	l	radial coordinate
t	$l[\rho(1-\nu^2)/E]^{1/2}$	time
p	$Eh/(1-\nu^2)l$	intensity of distributed load
w	l	plate transverse displacement
Q_r	$Eh/(1-\nu^2)l$	radial shear force
M_r, M_θ	D/l	radial and circumferential bending moments, respectively
β	l	radius of loading area
ψ	...	mean rotation angle of a line originally normal to median surface
κ^2	...	shear coefficient
ν	...	Poisson's ratio
$k^2 = [(1-\nu)/2]\kappa^2$	...	modified shear coefficient
$\alpha^2 = h^2/12l^2$	...	thickness parameter

Dimensional quantity	Physical interpretation
E	Young's modulus
h	plate thickness
$D = Eh^3/12(1-\nu^2)$	plate modulus
$G = E/2(1+\nu)$	shear modulus
$l = (D/\gamma)^{1/4}$	characteristic length
γ	specific weight of water (foundation modulus)
ρ	specific mass of plate (ice)

Introduction

THE problem of deformation and strength of floating ice sheets under normal loads has interested researchers for many years, and what appears to be the earliest rational analysis in this area can be traced to the classical work of H. Hertz[1] who treated the case of the static deformation of an infinite plate (ice sheet) on a linearly elastic (Winkler) foundation, subjected to a concentrated load. Subsequent analyses in this area treat the case of distributed loads acting on an infinite ice sheet[2] as well as the case of plates of bounded area under a variety of loading and boundary conditions.[3] Because the simple model of a Winkler foundation is often used to approximate the bending of slabs resting on soil, such calculations are usable for the case of floating ice plates and, in fact, represent a better model for this application since the resisting force due to buoyancy is rigorously represented by the Winkler model, i.e., resistance to deformation is directly proportional to transverse displacement in this case. Almost all the work concerned with floating ice sheets (or plates on elastic foundation) employs classical plate theory and assumes the existence of static conditions, although Refs. 4 and 5, while still treating the static case, utilize an improved plate theory that includes the effects of shear deformation.

Present-day operations in the polar regions as well as on inland lakes give rise to loading conditions on floating ice sheets which are clearly dynamic in character; such loadings result from aircraft landing and takeoff, air-drops of supplies, etc.[6] It is the purpose of the present investigation to predict the dynamic response of an ice sheet of infinite extent when subjected to a suddenly applied normal load. This solution also characterizes the initial response of an ice sheet of bounded extent, i.e., for a sufficiently small time interval between the instant of load application and the time when reflected waves from the boundaries first arrive in the solution domain of interest. The present analysis is carried out within the framework of an improved plate theory[7] that includes the effects of shear deformation and rotatory inertia as well as the effects of flexure and transverse inertia.

Formulation and Solution

The long-time, static behavior of flat ice plates (sheets) has been shown to be viscoelastic in character,[6] and because of the mechanism of ice formation it is known that floating ice plates are anisotropic. For the present case, however, we are concerned with the (materially) gross behavior under short-time, dynamic conditions. For these reasons it is assumed that the plate is elastic and obeys Hooke's law, and that the assumption of isotropy is adequate to obtain response estimates of sufficient accuracy. It should be noted, however, that viscoelastic effects as well as anisotropic behavior can be incorporated into the present type of analysis, although this would result in increased computational difficulties.

For the sake of generality as well as convenience, all subsequent equations are written in nondimensional form. Conversion to dimensional quantities is achieved with the help of the Nomenclature. In view of the foregoing, the equations

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characterizing the deformation of the floating ice sheet in axisymmetric polar coordinates are[7]

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\psi}{r^2} - \frac{k^2}{\alpha^2}\left(\frac{\partial w}{\partial r} + \psi\right) = \frac{\partial^2\psi}{\partial t^2} \quad (1a)$$

$$k^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial\psi}{\partial r} + \frac{\psi}{r}\right) - \alpha^2 w + p = \frac{\partial^2 w}{\partial t^2} \quad (1b)$$

The associated stress-displacement relations are given by

$$M_r = \partial\psi/\partial r + \nu\psi/r \quad (2a)$$

$$M_\theta = \psi/r + \nu\partial\psi/\partial r \quad (2b)$$

$$Q_r = k^2(\psi + \partial w/\partial r) \quad (2c)$$

and it is assumed that the plate is initially undeformed and at rest, i.e.,

$$w(r,0) = \dot{w}(r,0) = \psi(r,0) = \dot{\psi}(r,0) = 0 \quad (3)$$

In addition, it will be required that deformations satisfy the following conditions at infinity:

$$\lim_{r\to\infty}[rw(r,t)] = 0 \qquad \lim_{r\to\infty}[r\psi(r,t)] = 0 \quad (4)$$

At $t = 0$, a uniformly distributed load of circular platform (dimensionless radius β) is suddenly applied and remains in that position, i.e.,

$$\left.\begin{aligned} p(r,t) &= -p_0 H(t) \qquad 0 \le r \le \beta \\ p(r,t) &= 0 \qquad \beta < r \end{aligned}\right\} \quad (5)$$

where $H(t)$ is the unit step function.

We shall use the method of integral transforms to obtain a solution of the problem characterized by Eqs. (1–5). Thus the following Hankel transform pairs will be employed:

$$w^*(\xi,t) = \int_0^\infty rw(r,t)J_0(\xi r)dr \quad (6a)$$

$$w(r,t) = \int_0^\infty \xi w^*(\xi,t)J_0(\xi r)d\xi \quad (6b)$$

$$\psi^*(\xi,t) = \int_0^\infty r\psi(r,t)J_1(\xi r)dr \quad (7a)$$

$$\psi(r,t) = \int_0^\infty \xi\psi^*(\xi,t)J_1(\xi r)d\xi \quad (7b)$$

where J_0 and J_1, denote Bessel functions of the first kind, of order zero and one, respectively. The following relat'ons are readily established with the help of Eqs. (6) and (7), provided conditions expressed by Eqs. (4) hold:

$$\int_0^\infty r\frac{\partial w}{\partial r}J_1(\xi r)dr = -\xi w^*(\xi,t) \quad (8)$$

$$\int_0^\infty \frac{\partial}{\partial r}(r\psi)\cdot J_0(\xi r)dr = \xi\psi^*(\xi,t) \quad (9)$$

$$\int_0^\infty r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)w\cdot J_0(\xi r)dr = -\xi^2 w^*(\xi,t) \quad (10)$$

$$\int_0^\infty r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\psi\cdot J_1(\xi r)dr = -\xi^2\psi^*(\xi,t) \quad (11)$$

$$\int_0^\infty r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)^2 w\cdot J_0(\xi r)dr = \xi^4 w^*(\xi,t) \quad (12)$$

$$\int_0^\infty r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)^2 \psi\cdot J_1(\xi r)dr = \xi^4\psi^*(\xi,t) \quad (13)$$

We now multiply both sides of Eq. (1a) by the kernel $rJ_1(\xi r)$ and then integrate the equation with respect to r between the limits 0 and ∞. A similar operation is performed on Eq. (1b), except that in this case we use the kernel $rJ_0(\xi r)$. We then take the conventional Laplace transform of both equations, noting initial conditions as expressed by Eqs. (3). Utilizing Eqs. (6–11), the following doubly transformed set of equations is obtained:

$$\left.\begin{aligned} -\xi^2\overline{\psi^*} - (k^2/\alpha^2)(\overline{\psi^*} - \xi\overline{w^*}) &= s^2\overline{\psi^*} \\ k^2\xi(\overline{\psi^*} - \xi\overline{w^*}) - \alpha^2\overline{w^*} + \overline{p^*} &= s^2\overline{w^*} \end{aligned}\right\} \quad (14)$$

where bars denote Laplace transformed quantities and s is the Laplace transform parameter, and where [see Eq. (5)]

$$\overline{p^*}(\xi,s) = -\frac{p_0\beta J_1(\xi\beta)}{\xi s} \quad (15)$$

Solving, we obtain the doubly transformed solution

$$\overline{w^*}(\xi,s) = -\frac{p_0\beta(s^2 + a^2)J_1(\beta\xi)}{\xi s[(s^2 + a^2)(s^2 + b^2) - c^2]} \quad (16)$$

$$\overline{\psi^*}(\xi,s) = -\frac{p_0 k^2\beta J_1(\beta\xi)}{\alpha^2 s[(s^2 + a^2)(s^2 + b^2) - c^2]} \quad (17)$$

where

$$a^2 = \xi^2 + \frac{k^2}{\alpha^2} \qquad b^2 = k^2\xi^2 + \alpha^2 \qquad c^2 = \frac{k^4}{\alpha^2}\xi^2 \quad (18)$$

Taking inverse Laplace and Hankel transforms of Eqs. (16) and (17), we obtain the solution in the form of improper integrals,

$$w(r,t) = -\beta p_0 \int_0^\infty J_0(r\xi)J_1(\beta\xi)\frac{a^2}{a^2b^2 - c^2} \times \left\{1 - \frac{1}{2}(\cos dt + \cos et) - \frac{1}{2a^2}(a^2b^2 - a^4 - 2c^2) \times [(a^2 - b^2)^2 + 4c^2]^{-1/2}(\cos et - \cos dt)\right\} d\xi \quad (19)$$

$$\psi(r,t) = -\frac{k^2\beta p_0}{\alpha^2}\int_0^\infty \frac{\xi J_1(r\xi)J_1(\beta\xi)}{a^2b^2 - c^2} \times \left\{1 - \frac{1}{2}(\cos dt + \cos et) - \frac{a^2 + b^2}{2} \times [(a^2 - b^2)^2 + 4c^2]^{-1/2}(\cos dt - \cos et)\right\} d\xi \quad (20)$$

where

$$\left.\begin{matrix} d^2 \\ e^2 \end{matrix}\right. = \frac{a^2 + b^2}{2} \mp \left[\frac{(a^2 + b^2)^2}{4} - a^2b^2 + c^2\right]^{1/2}$$

Upon substitution of Eq. (20) into Eq. (2a), we obtain the radial bending moment

$$M_r(r,t) = -\frac{k^2\beta p_0}{\alpha^2}\int_0^\infty \frac{\xi^2 J_1(\beta\xi)}{a^2b^2 - c^2} \times \left[J_0(r\xi) - (1 - \nu)\frac{J_1(r\xi)}{r\xi}\right]\cdot\{1 - \tfrac{1}{2}(\cos dt + \cos et) - \tfrac{1}{2}(a^2 + b^2)[(a^2 - b^2)^2 + 4c^2]^{-1/2}(\cos dt - \cos et)\} d\xi \quad (21)$$

Table 1 Relation of thickness to characteristic length of ice sheet

h, in.	h/l
12	0.0542
120	0.0963
360	0.1267
600	0.1440
720	0.1508
1,200	0.1712
6,000	0.2560
60,000	0.4555

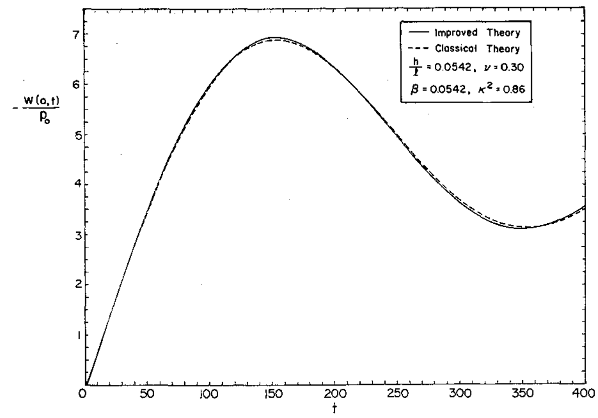

Fig. 1 Deflection at $r = 0$ vs time.

The present problem can also be solved, of course, within the framework of classical plate theory which neglects the effects of shear deformation and rotatory inertia. In this case, the basic equation of motion in axisymmetric polar

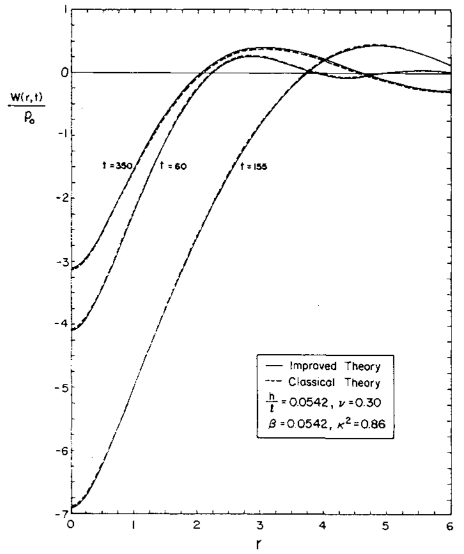

Fig. 2 Deflection vs r.

coordinates is[3]

$$\left\{\alpha^2\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]^2+\frac{\partial^2}{\partial t^2}+\alpha^2\right\}w = p \qquad (22)$$

and the applied load is again characterized by Eqs. (5). At $t = 0$ the plate is assumed to be undeformed and at rest.

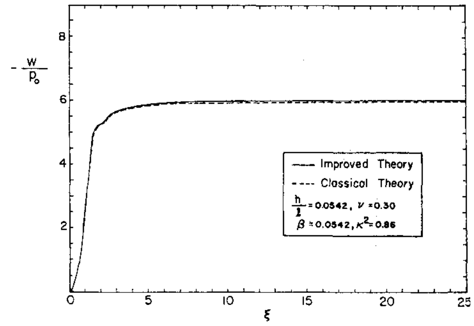

Fig. 3 Convergence of numerical integration for deflection w at $t = 100$, $r = 0$.

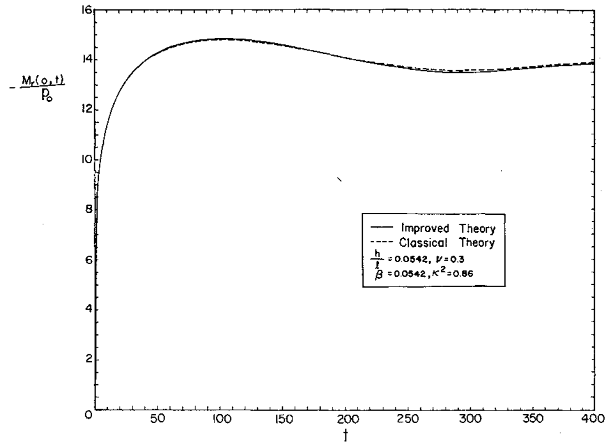

Fig. 4 Bending moment at $r = 0$ vs time.

Moreover, the deflection function is required to satisfy the first of Eqs. (4). We now multiply Eq. (22) by $rJ_0(\xi r)$ and integrate with respect to r from zero to infinity (i.e., we take the Hankel transform). If we then Laplace-transform this result, we obtain

$$\overline{w^*}(\xi,s) = \overline{p^*}(\xi,s)/[\alpha^2(\xi^4+1)+s^2] \qquad (23)$$

where $\overline{p^*}(\xi,s)$ is given by Eq. (15) and where we have used Eqs. (6a) and (12). Upon taking the inverse Hankel and Laplace transforms of Eq. (23), we obtain the solution

$$W(r,t) = -p_0\beta\int_0^\infty \frac{J_1(\xi\beta)J_0(\xi r)}{\alpha^2(\xi^4+1)}\,[1-\cos(\xi^4+1)^{1/2}\alpha t]d\xi \qquad (24)$$

and the corresponding bending moment is given by

$$M_r(r,t) = -p_0\beta\int_0^\infty \frac{\xi^2 J_1(\xi\beta)}{\alpha^2(\xi^4+1)}\times \left[J_0(\xi r)-(1-\nu)\frac{J_1(r\xi)}{r\xi}\right][1-\cos(\xi^4+1)^{1/2}\alpha t]d\xi \qquad (25)$$

Numerical Example

We now consider the following specific case, pertaining to an ice sheet floating on seawater:

$$\gamma = 0.0374 \text{ lb/in.}^3 \qquad E = 568{,}000 \text{ lb/in.}^2$$

$$\nu = 0.3 \qquad h = 12 \text{ in.} \qquad \kappa^2 = 0.86$$

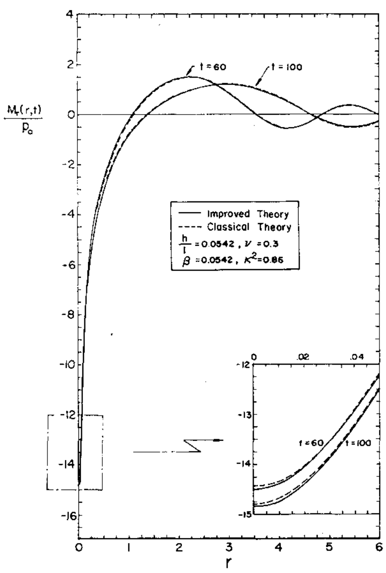

Fig. 5 Bending moment vs r.

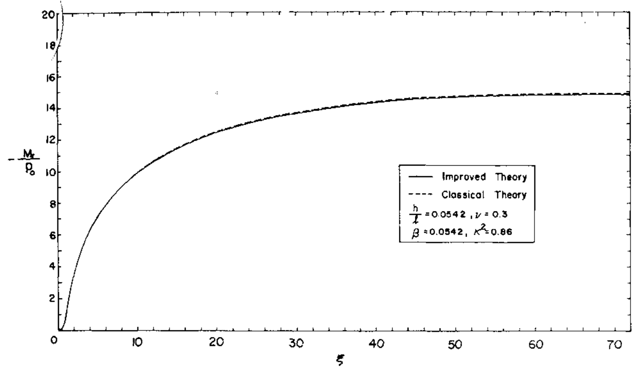

Fig. 6 Convergence of bending moment M_r at $t = 100$, $r = 0$.

Thus we have

$$D = Eh^3/12(1 - \nu^2) = 9 \times 10^7 \text{ lb-in.}$$

$$l = (D/\gamma)^{1/4} = 221.3 \text{ in.} \qquad h/l = 0.0542$$

If we take the radius of the loaded, circular area equal to the thickness of the ice sheet we have $\beta = h/l = 0.0542$. These numbers were substituted into Eqs. (19, 21, 24 and 25), and the required integrations were performed numerically on an IBM 7044 digital computer. Results of these computations are shown in Figs. 1–6.

Inspection of Figs. 1–6 indicates that results obtained within the framework of improved plate theory are almost identical to the results obtained by using classical plate theory for this special example. Because of this, the additional case of an ice sheet of thickness $h = 60$ ft, and $\beta = \frac{1}{10}h/l$, $\nu = 0.3$, and $\kappa^2 = 0.86$ was considered. The qualitative results of this computation are similar to the results of the previous example. The differences between classical and improved theory are somewhat more pronounced in the case of deflection (see Fig. 7). These differences again practically disappear for moments. It should be noted that it is possible to obtain greater differences between classical and improved plate theory results. Let us consider the relation of ice sheet thickness to characteristic length l. In Table 1 this relation is indicated for the presently used numerical values. It has been shown for the static case that differences between the two theories become more pronounced when plate thickness and characteristic length are of the same order of magnitude. In the present case we have $h/l = 0.0542$ and 0.1508, and therefore the difference between the solutions obtained by the two theories is negligible. However, because of practical considerations, in the case of very thick plates we would have a very small loading area, i.e., $\beta \ll h/l$. In this case, plate action would not predominate, and a meaningful picture of displacements and stresses would have to be provided by three-dimensional elasticity theory.

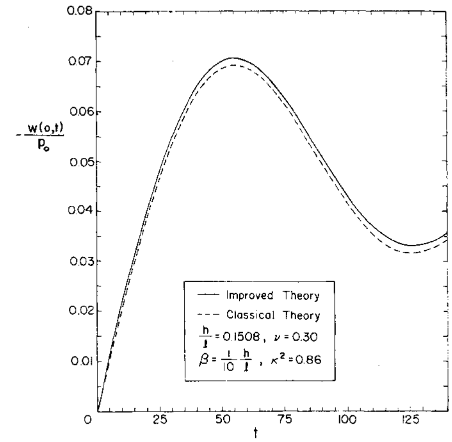

Fig. 7 Deflection at $r = 0$ vs time.

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